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Spherical vortex motions of a conducting fluid

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Some exact solutions of the steady M.H.D. equations for an inviscid infinitely conducting fluid are discussed, and a generalization of the Prendergast model for an idealized magnetic star is obtained, in which there is a finite azimuthal swirl velocity of the fluid. The flow in all examples is bounded by a sphere and expressions are found for the distribution of energy between the fluid motion and magnetic field.

Introduction

Studies of the motion of a highly ionized medium or infinitely conducting inviscid fluid in the presence of a magnetic field are of astrophysical interest in connexion with idealized models of magnetic stars. Ferraro (1954) found a general condition which must be satisfied by any poloidal magnetic field in equilibrium with an incompressible fluid, and gave the first-order calculations of the eccentricity of an equilibrium spheroid in which the leading term for the exterior field was that of a magnetic dipole. Roberts (1955) extended this work by obtaining a series expansion for the equation of the surface of a body with Ferraro's field. Lüst & Schlüter (1954) studied the case in which the magnetic field and electric current are everywhere parallel. Since such a field exerts no force, it can be imbedded in any conducting fluid and therefore, possesses the advantage that the pressure is constant over the surface. Prendergast (1956) generalized the force-free field solution of Lüst & Schlüter by obtaining a solution of the magneto-hydrostatic equations in which the magnetic field vanishes on the surface of a sphere, and the sphere is a surface of constant pressure. It is also to be noted that Chandrasekhar (1956*a*, *c*) gave explicit expressions for a large class of force-free fields. In all these configurations, the fluid is either at rest or rotating with constant angular velocity like a rigid body. There are few examples of flows in which the fluid velocity has a poloidal component. In one such example (studied by Chandrasekhar 1956*b*), the streamlines for the fluid motion and the magnetic lines of force coincide. This solution gives rise to an equipartition of energy densities between magnetic field and fluid motion, and has been shown to be a stable solution of the equation of motion.

In this paper some examples of flows, which are exact solutions of the equations of motion, are described for the case in which a finite motion of the fluid takes place in the presence of a magnetic field. The first flow discussed is an axially symmetric fluid motion (in a spherical container) in the presence of a toroidal magnetic field. The streamlines for the flow are similar to those of a Hill's spherical vortex, and nest about a stagnation point in the equatorial plane. If the magnetic field vanishes on the sphere the current density becomes infinite on the surface, but the total current contained by the sphere is finite, and there is a partition of the total energy in which the toroidal energy of the magnetic field is twice that of the fluid motion. If the region exterior to the sphere is non-conducting, and the magnetic field is discontinuous at the surface, there is a surface current sheet, and it is shown that the magnetic field energy is at least twice the energy of the fluid motion. An analogue of this solution is shown to exist for a finite swirl motion of the fluid in the presence of a poloidal magnetic field. These solutions are generalized by including the effects of a swirl motion of the fluid and a toroidal field respectively. This, of course, alters the distribution of energy between the magnetic field and fluid motion, and can be adjusted to give an equipartition of energy, which, from Chandrasekhar's work, suggests stability of the flow configuration. A variational principle is presented which characterizes these flows. Two other flow configurations are studied which have some relevance to idealized models of a magnetic star. The first of these is a general axially symmetric flow (poloidal and toroidal) in the presence of a toroidal magnetic field. Both the magnetic field and fluid velocity vanish on the sphere, so that the sphere is a surface of constant pressure and the motion inside the sphere is finite. The second flow is a generalization of the Prendergast (1956) model, in which a finite swirl motion of the fluid takes place in the presence of a general axially symmetric magnetic field. The magnetic field and fluid velocity vanish on the boundary so that the sphere is a surface of constant pressure and all the physical quantities of interest are finite inside the sphere.

Equations of motion

The equations of steady motion for an inviscid infinitely conducting fluid are

$$\left.\begin{aligned} \rho(\mathbf{q}\,.\,\nabla)\,\mathbf{q}+\mu[\mathbf{H}\wedge\operatorname{curl}\mathbf{H}] &= -\operatorname{grad}p,\\ \operatorname{curl}[\mathbf{q}\wedge\mathbf{H}] &= 0,\\ \operatorname{div}\mathbf{q} = 0, \quad \operatorname{div}\mathbf{H} &= 0, \end{aligned}\right\} \tag{1}$$

where $\mathbf{q}$ is the fluid velocity, $\mathbf{H}$ the magnetic field, p the pressure, μ the magnetic permeability, and ρ the density. Let (z, ω, ϕ) be cylindrical polar co-ordinates and consider a flow for which the fluid velocity is axially symmetric in the presence of an axially symmetric magnetic field. The velocity and magnetic fields may be written in the forms

$$\mathbf{q} = \operatorname{curl}\left\{\frac{-\psi}{\omega}\hat{\phi}\right\}+\frac{V}{\omega}\hat{\phi}, \quad H = \operatorname{curl}\left\{-\frac{\chi}{\omega}\hat{\phi}\right\}+\frac{U}{\omega}, \tag{2}$$

where $\hat{\phi}$ is the unit vector directed perpendicular to the azimuthal plane $\phi =$ constant, and in the sense of ϕ increasing. ψ is the Stokes stream function, χ is the flux function for the poloidal field and V/ω is the rotational or swirl component of the fluid velocity field. In terms of ψ, V, χ and U, the equations of motion are

$$\left.\begin{aligned}\frac{2\mu}{\rho}\frac{U}{\omega^2}\frac{\partial U}{\partial z}-\frac{2V}{\omega^2}\frac{\partial V}{\partial z}+\omega\frac{\partial\{\psi,L_{-1}(\psi)/\omega^2\}}{\partial(z,\omega)}-\frac{\mu}{\rho}\omega\frac{\partial\{\chi,L_{-1}(\chi)/\omega^2\}}{\partial(z,\omega)}&=0,\\ \omega\frac{\partial\{\psi,U/\omega^2\}}{\partial(z,\omega)}-\omega\frac{\partial\{\chi,V/\omega^2\}}{\partial(z,\omega)}&=0,\\ \frac{\partial(\psi,V)}{\partial(z,\omega)}=\frac{\mu}{\rho}\frac{\partial(\chi,U)}{\partial(z,\omega)},\quad \frac{\partial(\chi,\psi)}{\partial(z,\omega)}&=0,\end{aligned}\right\}\tag{3}$$

where the Stokes operator L_{-1} is defined by

$$L_{-1}\equiv\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial\omega^2}-\frac{1}{\omega}\frac{\partial}{\partial\omega}.$$

These equations have been given in a slightly different form by Chandrasekhar (1956) and Lüst & Schlüter (1954) for the more general case in which diffusion terms are included. The pressure is determined from the equations

$$\frac{\mu}{\rho}\frac{1}{\omega^2}\frac{\partial\chi}{\partial z}L_{-1}(\chi)-\frac{1}{\omega^2}\frac{\partial\psi}{\partial z}L_{-1}(\psi)+\frac{\mu}{\rho}\frac{U}{\omega^2}\frac{\partial U}{\partial z}-\frac{V}{\omega^2}\frac{\partial V}{\partial z}=-\frac{\partial P}{\partial z},$$

where
$$P=\frac{p}{\rho}+\tfrac{1}{2}|\mathbf{q}|^2.$$

It is to be noted that there is a solution of (1) given by $\mathbf{q}=(\mu/\rho)^{\frac{1}{2}}\mathbf{H}$, and the stability of this solution has been investigated by Chandrasekhar (1956*b*).

Flow without swirl in the presence of a toroidal field

In this case $V=\chi=0$, and the equations of motion are

$$\frac{2\mu}{\rho}\frac{U}{\omega^2}\frac{\partial U}{\partial z}+\omega\frac{\partial\{\psi,L_{-1}(\psi)/\omega^2\}}{\partial(z,\omega)}=0,\tag{4}$$

$$\frac{\partial\{\psi,U/\omega^2\}}{\partial(z,\omega)}=0;\tag{5}$$

and it is to be noted that the magnetic field is orthogonal to the fluid velocity. Equation (5) implies, in general, that

$$U=\omega^2 f(\psi),\tag{6}$$

where f is an arbitrary function, so that, if (6) is substituted in (4), the former equation is of the form

$$\frac{2\mu}{\rho}\omega^2 f(\psi)f'(\psi)\frac{\partial\psi}{\partial z}+\omega\frac{\partial\{\psi,L_{-1}(\psi)/\omega^2\}}{\partial(z,\omega)}=0,\tag{7}$$

which may be written as

$$\frac{\partial\{\psi,G/\omega^2\}}{\partial(z,\omega)}=0,\tag{8}$$

where
$$G\equiv L_{-1}(\psi)+\frac{\mu}{\rho}\omega^4 f(\psi)f'(\psi).\tag{9}$$

Thus, in general,

$$G = L_{-1}(\psi) + \frac{\mu}{\rho}\omega^4 f(\psi) f'(\psi) = \omega^2 F(\psi), \tag{10}$$

where F is an arbitrary function. The pressure is given by

$$p + \tfrac{1}{2}\rho\,|\mathrm{curl}\,(-\psi/\omega)\,\hat{\phi}|^2 = \mu\omega^2 f^2(\psi) - \rho\int F(\psi)\,d\psi. \tag{11}$$

Some special forms for f and F are now considered which permit solutions of (10). Set

$$F(\psi) = 0, \quad f(\psi)f'(\psi) = \kappa, \tag{12}$$

where κ is a positive constant, so that

$$f^2(\psi) = 2K\psi + a_1, \tag{13}$$

where a_1 is a constant, and (10) reduces to

$$L_{-1}(\psi) = -\left(\frac{K\mu}{\rho}\right)\omega^4, \tag{14}$$

for which a general solution is

$$\psi = -\frac{K\mu\omega^6}{24\rho} + \psi^{(-1)}, \tag{15}$$

where $\psi^{(-1)}$ is a general solution of $L_{-1}\{\psi^{(-1)}\} = 0$. If spherical polar co-ordinates (r, θ) are defined by $z = r\cos\theta$, $\omega = r\sin\theta$, $\beta = \cos\theta$, then (14) may be rewritten as

$$\left\{\frac{\partial^2}{\partial r^2} + \frac{1-\beta^2}{r^2}\frac{\partial^2}{\partial\beta^2}\right\}\psi = -\frac{K\mu}{\rho} r^4(1-\beta^2)\{4-(5\beta^2-1)\}. \tag{16}$$

Now (16) yields solutions which are regular in a simply connected domain, and the simplest boundary to consider is a sphere $r = a$. In this case, an appropriate form for ψ is given by

$$\psi = F_1(r)(1-\beta^2)(5\beta^2-1) + F_2(r)(1-\beta^2), \tag{17}$$

and the equations to be satisfied by F_1 and F_2 are

$$F_1'' - \frac{12F_1}{r^2} = \frac{K\mu}{5\rho}r^4, \quad F_2'' - \frac{2F_2}{r^2} = -\frac{4K\mu}{5\rho}r^4. \tag{18}$$

If the sphere is a streamline, then $F(a) = G(a) = 0$, and, assuming the motion is finite inside the sphere, the required solutions are

$$F_1(r) = \frac{K\mu}{90\rho}(r^6 - a^2r^4), \quad F_2(r) = \frac{K\mu}{35\rho}(a^4r^2 - r^6); \tag{19}$$

ψ may be written as

$$\psi = \frac{K\mu}{630\rho}(a^2-r^2)r^2(1-\beta^2)\{18(a^2-r^2) + 8r^2 + 35r^2(1-\beta^2)\}, \tag{20}$$

which is clearly positive inside the sphere $r = a$. Set $\lambda = r/a$, $A = K\mu a^6/630\rho$. Then (20) is replaced by

$$\psi = A\lambda^2(1-\lambda^2)[18 - 10\lambda^2 + 35(1-\beta^2)](1-\beta^2). \tag{21}$$

The components of fluid velocity are

$$u = \frac{1}{a^2\lambda^2}\frac{\partial\psi}{\partial\beta}, \quad v = \frac{1}{a^2\lambda\sin\theta}\frac{\partial\psi}{\partial\lambda}. \tag{22}$$

The radial component u vanishes at $\beta = 0$, or $\theta = \frac{1}{2}\pi$ (that is, on the equatorial plane and also at $r = a$, since the surface is a streamline). The tangential component v vanishes at $\sin\theta = 0$ (that is, $\theta = 0$), so the poles are stagnation points for the flow. v also vanishes for

$$75\lambda^4 - 14\lambda^2 - 18 = 0, \tag{23}$$

or, equivalently,

$$\lambda^2 = \frac{7 \pm \sqrt{1399}}{75}. \tag{24}$$

The negative sign is inadmissible, so that there is a stagnation line in the equatorial plane given by

$$r_0 = a\left\{\frac{7+\sqrt{1399}}{75}\right\}^{\frac{1}{2}}, \quad \theta = \tfrac{1}{2}\pi. \tag{25}$$

It is readily verified that on $\theta = \frac{1}{2}\pi$, v is positive for $0 \leqslant r < r_0$ and negative for $r_0 < r \leqslant a$, so that the fluid rotates in the azimuthal plane in closed streamlines about the stagnation point inside the sphere. Now, it is supposed that the region exterior to the sphere is non-conducting, and the exterior magnetic field vanishes in the region $r > a$, since the only non-zero toroidal field, satisfying the equations

$$\operatorname{curl}\mathbf{H} = 0, \quad \operatorname{div}\mathbf{H} = 0, \tag{26}$$

is singular on the axis. The magnetic field is continuous on the sphere $r = a$ if $a_1 = 0$ in (13), and is then given by

$$\mathbf{H} = (2K\psi)^{\frac{1}{2}}\,\omega\hat{\phi}. \tag{27}$$

The current density $\mathbf{j}$ is defined by

$$\mathbf{j} = \operatorname{curl}\mathbf{H} = \frac{1}{r^2}\frac{\partial}{\partial\beta}[(2K\psi)^{\frac{1}{2}}r^2(1-\beta^2)]\,\hat{r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial r}[(2K\psi)^{\frac{1}{2}}r^2(1-\beta^2)]\,\hat{\theta}, \tag{28}$$

so that $(\mathbf{j}\,.\,\hat{r}) = 0$ on $r = a$, and the tangential component $(\mathbf{j}\,.\,\hat{\theta})$ becomes infinite like $(a-r)^{-\frac{1}{2}}$ on $r = a$, but the total current inside the sphere is zero. If $a_1 > 0$, the (tangential) magnetic field is discontinuous at $r = a$ and there must be a surface current sheet on the boundary of the sphere. In this case, the magnetic field inside the sphere is given by

$$\mathbf{H} = \omega\{2K\psi + a_1\}^{\frac{1}{2}}\,\hat{\phi}, \tag{29}$$

and the current density is given by (28), in which $2K\psi$ is replaced by $2K\psi + a_1$. The pressure distribution is determined from the equation,

$$p + \tfrac{1}{2}\rho q^2 = 2\mu K\psi + \mu a_1\omega^2 + \text{constant}, \tag{30}$$

so that in the case $a_1 = 0$, the total head of pressure is a constant on the sphere, while the pressure is proportional to the square of the tangential velocity $(1/r\sin\theta)(\partial\psi/\partial r)$. Now, the energies of the fluid motion and magnetic field are given by

$$T_L = \tfrac{1}{2}\rho\int_v |\mathbf{q}|^2\,dv, \quad T_M = \tfrac{1}{2}\mu\int_v |\mathbf{H}|^2\,dv, \tag{31}$$

where the integration is carried out over the region bounded by the sphere. If $a_1 = 0$, then it is found that

$$2T_L = T_M = \mu K \int_v \psi \omega^2 \, dv, \tag{32}$$

so there is a partition of the total energy in which the magnetic field energy is twice the kinetic energy of the fluid motion. For the case in which $a_1 > 0$, T_M is replaced by

$$T_M = \mu K \int_v \psi \omega^2 \, dv + \frac{\mu a_1}{2} \int \omega^2 \, dv, \tag{33}$$

so that $T_M > 2T_L$, or the magnetic field energy is at least twice as large as the kinetic energy of the fluid motion. These results are valid for any axially symmetric closed surface in which ψ satisfies (14) and the relevant boundary conditions on S.

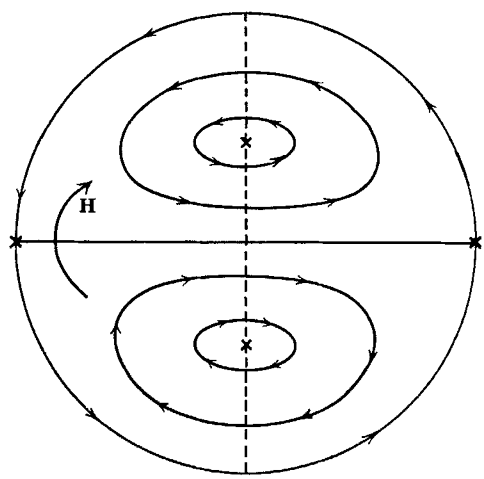

FIGURE 1. Diagram of the spherical vortex streamlines showing stagnation points on the equator and at the poles.

By setting $F(\psi) = -\alpha, \alpha > 0$, a slight generalization of (20) is obtained, and the stream function satisfying the boundary conditions is

$$\psi = \frac{K\mu}{630\rho}(a^2 - r^2) r^2(1-\beta^2)[18(a^2-r^2) + 8r^2 + 35r^2(1-\beta^2)] + \tfrac{1}{10}\alpha(a^2-r^2) r^2(1-\beta^2). \tag{34}$$

The second term is that for the interior motion of a Hill's spherical vortex, and the flow is qualitatively similar to the case $\alpha = 0$. Again, a linear partial differential equation for ψ is also obtained by setting $f(\psi) = \gamma\psi$, where γ is a constant and

$$\mathbf{H} = \gamma \omega \psi \hat{\phi}, \tag{35}$$

where ψ satisfies the equation

$$(L_{-1} + \gamma^2[\mu/\rho]\,\omega^4)\,\psi = 0. \tag{36}$$

There are no separable solutions of this equation in spherical co-ordinates, and a solution will not be attempted here. However, it might be mentioned that, if

$\psi = 0$ on S, then there is an equipartition of energy between the magnetic field and fluid motion, in which

$$T_L = T_M = \tfrac{1}{2}\mu\alpha^2 \int_v \psi^2\omega^2\,dv, \tag{37}$$

and, in addition, all the physical quantities, pressure, velocity, current density are finite inside S. Thus, a solution of (36) with $\psi = 0$ on S may be a stable solution of the equations of motion for an axially symmetric fluid motion in the presence of a toroidal magnetic field. The results in this section may be generalized by observing that, if $f(\psi) = K\psi^{1/n}$, then ψ satisfies

$$L_{-1}(\psi) + \frac{\mu}{n\rho} K^2\omega^4\psi^{(2/n)-1} = 0 \tag{38}$$

and $T_M = nT_L$. However, it is not clear for what values of n (38) yields solutions which are finite inside S. Finally, it is pointed out that solutions of

$$L_{-1}(\psi) + (\mu/\rho)\,\omega^4 f(\psi) f'(\psi) = 0 \tag{39}$$

yield stationary values of the integral,

$$T_M - T_L = \tfrac{1}{2}\rho \int_v \{(1/\omega^2)(\psi_\omega^2 + \psi_z^2) - (\mu/\rho)\,\omega^2 f^2(\psi)\}\,dv. \tag{40}$$

Flow with azimuthal swirl

If azimuthal swirl is added to the fluid motion, the equations of motion are

$$\left.\begin{aligned} \frac{2\mu}{\rho\omega^2} U\frac{\partial U}{\partial z} - \frac{2}{\omega^2}\frac{V\,\partial V}{\partial z} + \omega\frac{\partial\{\psi, L_{-1}(\psi)/\omega^2\}}{\partial(z,\omega)} &= 0,\\ \frac{\partial\{\psi, U/\omega^2\}}{\partial(z,\omega)} = 0, \quad \frac{\partial(\psi, V)}{\partial(z,\omega)} &= 0. \end{aligned}\right\} \tag{41}$$

Thus, in general, $U = \omega^2 f(\psi)$, $V = g(\psi)$, and the equation for ψ is

$$G \equiv L_{-1}(\psi) + (\mu/\rho)\,\omega^4 f(\psi) f'(\psi) + g(\psi)\,g'(\psi) = \omega^2 F(\psi). \tag{42}$$

As before, write $f^2(\psi) = 2K\psi + a_1$, and set $g(\psi) = \alpha\psi$, $F = 0$, where α is a constant. Then (42) reduces to

$$L_{-1}(\psi) + \alpha^2\psi = (\mu/\rho)\,K\omega^4. \tag{43}$$

A general solution is given by

$$\psi = -\frac{K\mu}{\rho\alpha^2}\omega^4 + \frac{8K\mu}{\rho\alpha^4}\omega^2 + \phi, \tag{44}$$

where ϕ is a general solution of

$$(L_{-1} + \alpha^2)\,\phi = 0. \tag{45}$$

An appropriate solution of (44) is

$$\chi^1 = Ar^{\frac{1}{2}}J_{\frac{3}{2}}(\alpha r)\,(1-\beta^2) + Br^{\frac{1}{2}}J_{\frac{7}{2}}(\alpha r)\,(5\beta^2 - 1)\,(1-\beta^2), \tag{46}$$

where $J_{n+\frac{1}{2}}(\alpha r)$, $n = 1, 3$, is the Bessel function of fractional order. If $\psi = 0$ vanishes on $r = a$, the constants A and B are given by

$$\left.\begin{aligned} Aa^{\frac{1}{2}}J_{\frac{3}{2}}(\alpha a) + \frac{8K\mu a^2}{\rho\alpha^4} - \frac{4K\mu a^4}{5\rho\alpha^2} &= 0, \\ Ba^{\frac{1}{2}}J_{\frac{7}{2}}(\alpha a) + \frac{K\mu}{5\rho\alpha^2} &= 0. \end{aligned}\right\} \tag{47}$$

The stream function is then given by

$$\begin{aligned} \psi = \left[\left\{\frac{8Kr^2}{\rho\alpha^2} - \frac{4K\mu r^4}{\rho\alpha^2}\right\} - \frac{r^{\frac{1}{2}}}{a}\frac{J_{\frac{3}{2}}(\alpha r)}{J_{\frac{3}{2}}(\alpha a)}\left\{\frac{8K\mu a^2}{\rho\alpha^4} - \frac{4}{5}\frac{K\mu a^4}{\rho\alpha^2}\right\}\right](1-\beta^2) \\ + (1-\beta^2)(5\beta^2-1)\frac{\mu}{5\rho\alpha^2}\left\{1 - \frac{r^{\frac{1}{2}}}{a}\frac{J_{\frac{7}{2}}(\alpha r)}{J_{\frac{7}{2}}(\alpha a)}\right\}, \end{aligned} \tag{48}$$

where it is supposed $J_{\frac{3}{2}}(\alpha a)$, $J_{\frac{7}{2}}(\alpha a) \neq 0$. If $a_1 > 0$, then the fields, vorticity, pressure and current are finite inside the sphere. For motion inside an axially symmetric closed surface S containing a region v, the kinetic energy and magnetic field energy are readily shown to be

$$\left.\begin{aligned} T_L &= \tfrac{1}{2}\mu K\int_v \psi\omega^2\,dv + \rho\alpha^2\int_v \frac{\psi^2}{\omega^2}\,dv, \\ T_M &= \mu K\int_v \psi\omega^2\,dv + \tfrac{1}{2}\mu a_1\int_v \omega^2\,dv. \end{aligned}\right\} \tag{49}$$

These expressions are valid for any motion satisfying (53) and the boundary condition $\psi = 0$ on S. The kinetic energy of the fluid motion is clearly increased by the rotation or swirl of the fluid, and an equipartition of the total energy exists if

$$\frac{a_1\mu}{2}\int_v \omega^2\,dv = \rho\alpha^2\int_v \frac{\psi^2}{\omega^2}\,dv - \tfrac{1}{2}\mu K\int_v \psi\omega^2\,dv. \tag{50}$$

By setting $F(\psi) = K_1$ a constant, the equation satisfied by ψ is

$$L_1(\psi) + a^2\psi = -\frac{\mu}{\rho}K\omega^4 + K_1\omega^2. \tag{51}$$

This equation is a generalization of the equation (43) for the case $K = 0$. If K_1 is regarded as an arbitrary constant, the general solution is

$$\psi = -\frac{\mu K}{\rho\alpha^2}\omega^4 + A\omega^2 + \phi, \tag{52}$$

where A is a constant related to K_1 by $K_1 = A\alpha^2 - 8\mu K/\rho\alpha^2$, and ϕ is a general solution of (45). In the present case, a suitable solution for ψ is

$$\begin{aligned} \psi = \left\{\frac{\mu K}{5\rho\alpha^2}r^4 + Cr^{\frac{1}{2}}J_{\frac{7}{2}}(\alpha r)\right\}(5\beta^2-1)(1-\beta^2) \\ + \left\{Ar^2 + Br^{\frac{1}{2}}J_{\frac{3}{2}}(\alpha r) - \tfrac{4}{5}r^4\frac{\mu K}{\rho\alpha^2}\right\}(1-\beta^2). \end{aligned} \tag{53}$$

A solution is sought which satisfies the condition of zero velocity on the sphere, so that the boundary conditions reduce to

$$\left.\begin{aligned} \frac{\mu K a^4}{5\rho\alpha^2} + C a^{\frac{1}{2}} J_{\frac{7}{2}}(\alpha a) &= 0, \\ \frac{4\mu K a^4}{5\rho\alpha^2} + C a^{\frac{1}{2}}\{\tfrac{1}{2} J_{\frac{7}{2}}(\alpha a) + \alpha a J'_{\frac{7}{2}}(\alpha a)\} &= 0, \\ A a^2 B a^{\frac{1}{2}} J_{\frac{3}{2}}(\alpha a) - \frac{4}{5}\frac{\mu K a^4}{\rho\alpha^2} &= 0, \\ 2Aa^2 + Ba^{\frac{1}{2}}\, \tfrac{1}{2} J_{\frac{3}{2}}(\alpha a) + \alpha a J'_{\frac{3}{2}}(\alpha a) - \frac{16 a^4 \mu K}{5\rho\alpha^2} &= 0. \end{aligned}\right\} \tag{54}$$

The constants A, B and C are found to be

$$\left.\begin{aligned} A &= \frac{a^2 \mu K}{\rho\alpha^2}\left[\frac{4}{5} + \frac{16}{3J_{\frac{3}{2}}(\alpha a) - 2\alpha a J'_{\frac{3}{2}}(\alpha a)}\right], \\ B &= -\frac{16 a^4 \mu K}{a^{\frac{1}{2}}\rho\alpha^2} \cdot \frac{1}{3J_{\frac{3}{2}}(\alpha a) - 2\alpha a J'_{\frac{3}{2}}(\alpha a)}, \\ C &= -\frac{\mu K a^4}{5\rho\alpha^2 \alpha^{\frac{1}{2}} J_{\frac{7}{2}}(\alpha a)}, \end{aligned}\right\} \tag{55}$$

provided that αa is a root of the equation,

$$7J_{\frac{7}{2}}(\alpha a) - 2\alpha a J'_{\frac{7}{2}}(\alpha a) = 0. \tag{56}$$

Thus, (53) represents an exact solution of the magnetohydrodynamic equations for an infinitely conducting fluid sphere in the presence of a toroidal magnetic field. If $a_1 = 0$, the magnetic field vanishes on the boundary, and the current is finite for $r \leqslant a$. The pressure is now given by

$$\frac{p}{\rho} + \tfrac{1}{2}|q|^2 = \frac{\mu}{\rho} K_1 \psi - \frac{K\mu}{\rho}\,\omega^2\psi + \text{constant}. \tag{57}$$

Also of interest is the fact that the fluid velocity vanishes on the sphere, which implies that the sphere is a surface of constant pressure.

The kinetic energy of the fluid motion is now given by

$$T_L = \tfrac{1}{2}\mu K \int_v \psi\omega^2\,dv + \rho\alpha \int_v \frac{\psi^2}{\omega^2}\,dv - \tfrac{1}{2}\rho K_1 \int_v \psi\,dv. \tag{58}$$

Flow with a poloidal field

In this case, the equations of motion are very nearly the same if $\chi = \alpha\psi$, where α is a constant, then the equation satisfied by ψ is

$$\frac{2\mu}{\rho}\,\omega^2 f(\psi) f'(\psi)\frac{\partial\psi}{\partial z} + \left(1 - \alpha^2\frac{\mu}{\rho}\right)\omega\,\frac{\partial\{\psi, L_{-1}(\psi)/\omega^2\}}{\partial(z,\omega)} = 0, \tag{59}$$

which, in general, implies that

$$\left(1 - \frac{\alpha^2\mu}{\rho}\right) L_{-1}(\psi) + \omega^4\frac{\mu}{\rho} f(\psi) f'(\psi) = \omega^2 F(\psi). \tag{60}$$

Set $F = 0, f^2(\psi) = 2K\psi$, then the stream function is given by

$$\psi = \frac{\mu}{630(\rho-\alpha^2\mu)}(a^2-r^2)r^2(1-\beta^2)\{18a^2-25r^2-35r^2\beta^2\}, \tag{61}$$

where $\rho \neq \alpha^2\mu$. In this motion, the fluid behaves in a similar manner to the previous flows, but takes place in the presence of both a poloidal and toroidal magnetic field. The total energy is $T = T_L + T_M$, where

$$T_L = \frac{1}{2}\frac{\mu\rho}{\rho-\mu\alpha^2}\int_v \psi\omega^2\,dv, \quad T_M = \left[\mu K + \frac{1}{2}\frac{\mu K\alpha^2\mu}{\rho-\alpha^2\mu}\right]\int_v \psi\omega^2\,dv. \tag{62}$$

Azimuthal swirl of the fluid in the presence of a poloidal and toroidal magnetic field

In this case, $\psi = 0$ in (3), and the equations of motion reduce to

$$\left.\begin{aligned} \frac{2\mu}{\rho}\frac{U}{\omega^2}\frac{\partial U}{\partial z} - \frac{2}{\omega^2}V\frac{\partial V}{\partial z} - \frac{\mu}{\rho}\omega\frac{\partial\{\chi, L_{-1}(\chi)/\omega^2\}}{\partial(z,\omega)} = 0, \\ \frac{\partial(\chi, V/\omega^2)}{\partial(z,\omega)} = 0, \quad \frac{\partial(\chi, U)}{\partial(z,\omega)} = 0. \end{aligned}\right\} \tag{63}$$

The latter two equations imply that

$$V = \omega^2 f(\chi), \quad U = g(\chi), \tag{64}$$

where f and g are arbitrary functions. The equation satisfied by χ is then

$$\frac{\mu}{\rho}\omega\frac{\partial\{\psi, L_{-1}(\chi)/\omega^2\}}{\partial(z,\omega)} + 2\omega^2 f(\chi)f'(\chi)\frac{\partial\chi}{\partial z} - \frac{2\mu}{\rho\omega^2}g(\chi)g'(\chi)\frac{\partial\chi}{\partial z} = 0, \tag{65}$$

which implies, in general, that

$$L_{-1}(\chi) + (\rho/\mu)\,\omega^4 f(\chi)f'(\chi) + g(\chi)g'(\chi) = \omega^2 F(\chi), \tag{66}$$

where F is arbitrary. First set $F = 0$, $f(\chi)f'(\chi) = \mu K/\rho$, $g(\chi) = \alpha\chi$. Then χ satisfies

$$L_{-1}(\chi) + \alpha^2\chi = -K\omega^4, \tag{67}$$

for which a general solution is

$$\chi = -\frac{K\omega^4}{\alpha^2} + \frac{8K\omega^2}{\alpha^4} + \phi, \tag{68}$$

where ϕ is a general solution of (55). For the case of a sphere, an appropriate solution is

$$\chi = Ar^{\frac{1}{2}}J_{\frac{7}{2}}(\alpha r)(5\beta^2-1)(1-\beta^2) + Br^{\frac{1}{2}}J_{\frac{3}{2}}(\alpha r)(1-\beta^2), \tag{69}$$

where A and B are arbitrary constants. The region exterior to the sphere is supposed non-conducting; then the boundary condition on the magnetic field is $(\mathbf{j}.\hat{r}) = (\operatorname{curl}\mathbf{H}.\hat{r}) = 0$, $r = a$. This is equivalent to $\chi = 0$, $r = a$, which also implies that $(\mathbf{H}.\hat{r}) = 0$, $r = a$ (that is, the normal field vanishes on the sphere). The constants A and B are found on calculation to be given by

$$A = -\frac{Ka^4}{5\alpha^2 a^{\frac{1}{2}}J_{\frac{7}{2}}(\alpha a)}, \quad B = \frac{4Ka^4\alpha^2 - 40Ka^2}{5\alpha^2 a^{\frac{1}{2}}J_{\frac{3}{2}}(\alpha a)}. \tag{70}$$

Now $f^2(\chi) = (2\mu/\rho)K\chi + a_0^2$, where a_0 is a constant and the fluid velocity is given by

$$\mathbf{q} = \omega\left\{\frac{2\mu}{\rho}K\chi + a_0^2\right\}^{\frac{1}{2}}\hat{\phi}. \tag{71}$$

So the fluid on the surface of the sphere is rotating with constant angular velocity a_0, since $\chi = 0$, $r = a$. If $a_0 = 0$, the sphere is a surface of constant pressure. The current density is finite everywhere inside the sphere.

The case $\alpha = 0$

In this situation, there is no toroidal magnetic field, and

$$L_{-1}(\chi) = -K\omega^4. \tag{72}$$

The solution for χ satisfying the boundary condition is

$$\chi = (K/630)(a^2-r^2)r^2(1-\beta^2)\{18(a^2-r^2)+8r^2+35r^2(1-\beta^2)\}. \tag{73}$$

The magnetic field has a neutral point at $r = r_0$, $\beta = 0$; and the magnetic lines of force are those of the spherical vortex described in the previous sections. It is of interest to note that the roles of the fluid and magnetic field are effectively reversed in this type of flow, since there is a partition of the energy in which

$$T_L = 2T_M = K\int_v \chi\omega^2\,dv; \tag{74}$$

that is, the kinetic energy of the fluid is twice that of the magnetic field. Equation (74) is valid only for the case $a_0 = 0$, so that the fluid velocity vanishes on the surface, but the vorticity of the fluid is infinite on the surface like $(a-r)^{-\frac{1}{2}}$ as $r \to a$. The pressure in the fluid is given by

$$P = \frac{p}{\rho}+\tfrac{1}{2}|\mathbf{q}|^2 = \frac{2\mu\omega^2}{\rho}K\chi + \text{constant}, \tag{75}$$

so that the sphere is a surface of constant pressure. By setting $F(\chi) = A_1$ a constant, the equation for χ may be extended to the form

$$(L_{-1}+\alpha^2)\chi = -K\omega^4 + A_1\omega^2. \tag{76}$$

This case has been investigated by Prendergast (1956) for the case $V = 0$ (no azimuthal velocity field). The advantage in considering solutions is that it is possible to find a true equilibrium configuration for the sphere in which the magnetic field vanishes on the surface and the sphere is a surface of constant pressure. A suitable solution of (76) is

$$\chi = \left\{Ar^2 - \frac{4Kr^4}{5\alpha^2} + Br^{\frac{1}{2}}J_{\frac{3}{2}}(\alpha r)\right\}(1-\beta^2) + \left\{\frac{Kr^4}{5\alpha^2} + Cr^{\frac{1}{2}}J_{\frac{7}{2}}(\alpha r)\right\}(5\beta^2-1)(1-\beta^2), \tag{77}$$

where A, B, C are arbitrary constants. The magnetic field on $r = a$ vanishes if $\chi = \partial\chi/\partial r = 0$, $r = a$. The constants A, B and C are then given by

$$A = \frac{4Ka^4}{5\alpha^2} - \frac{8Ka^4}{5\alpha^2\{\alpha a J'_{\frac{3}{2}}(\alpha a) - \frac{3}{2}J_{\frac{3}{2}}(\alpha a)\}}, \tag{78}$$

$$B = \frac{8Ka^4}{5\alpha^2\{\alpha a^{\frac{3}{2}}J'_{\frac{3}{2}}(\alpha a) - \frac{3}{2}a^{\frac{1}{2}}J_{\frac{3}{2}}(\alpha a)\}}, \tag{79}$$

$$C = -\frac{Ka^4}{5\alpha^2 a^{\frac{1}{2}}J_{\frac{7}{2}}(\alpha a)},$$

where αa is a root of the equation,

$$7J_{\frac{7}{2}}(\alpha a) - 2\alpha a J'_{\frac{7}{2}}(\alpha a) = 0. \tag{80}$$

The exterior of the sphere is supposed non-conducting, and the field is zero for $r > a$, and is therefore continuous at the surface. The pressure is given by

$$P = \frac{p}{\rho} + \tfrac{1}{2}|\mathbf{q}|^2 = \frac{2\mu K\omega^2\chi}{\rho} - \frac{\mu A_1}{\rho}\chi + \text{constant}, \tag{81}$$

so that if $a_0 = 0$, the fluid velocity vanishes on the surface and the configuration is one of true equilibrium, since p is a constant over $r = a$. Both the vorticity and current are finite over the interior of the sphere, and the boundary condition $(\mathbf{j}.\hat{r}) = 0$ is satisfied at the surface $r = a$. If $a_0 \neq 0$, then the configuration corresponds to one of quasi-equilibrium, since p does not vanish at the surface. Also, in this case the boundary is rotating with constant angular velocity a_0. Thus, (87) may be regarded as a generalization of the true equilibrium configuration for a fluid sphere found by Prendergast, in which the additional effect of azimuthal swirl of the fluid has been included. The kinetic energy of the fluid is given by

$$T_L = \mu \int_v \chi\omega^2\, dv, \tag{82}$$

and the magnetic energy is

$$T_M = \mu\alpha^2 \int_v \frac{\chi^2}{\omega^2}\, dv + \tfrac{1}{2}K\mu \int_v \chi\omega^2\, dv - \tfrac{1}{2}A_1 \int_v \chi\, dv, \tag{83}$$

where $A = A_1/\alpha^2 + 8K/\alpha^4$.

Other equilibrium configurations

Analogous to (35) and (36), set $f(\chi) = (\mu/\rho)^{\frac{1}{2}}\alpha\chi$, $g = R = 0$, then χ satisfies

$$L_{-1}(\chi) + \alpha^2\omega^4\chi = 0, \tag{84}$$

and the boundary conditions are satisfied if $\chi = 0$, $r = a$. There is an equipartition of energy between the fluid motion and magnetic field energy, but, due to the fact that (84) does not possess separable solutions in spherical polar coordinates, it is not possible to determine the solution for a sphere.

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